Cauchy integral

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Reminder (Taylor's Thm) $\sum a_n w^n$ is infinitely differentiable for $|w| < R$, (R - radius of convergence).

$$f^{(k)}(w) = \sum_{n=k}^{\infty} \frac{n!}{(n-k)!} a_n w^{n-k}$$

$$a_n = \frac{f^{(n)}(0)}{n!}$$

Theorem Let γ be a (piecewise smooth) curve, φ - piecewise continuous bounded function on γ.

For $z \notin \gamma$, let $F(z) := \oint_\gamma \frac{\varphi(\zeta)}{\zeta - z} d\zeta$.

Then $F \in \mathcal{A}(\mathbb{C} \setminus \gamma)$.

Moreover, if $z_0 \notin \gamma$ and $R = \operatorname{dist}(z_0, \gamma) = \inf_{\zeta \in \gamma} |\zeta - z_0|$.

then for $|z - z_0| < R$,

$$F(z) = \sum_{n=0}^{\infty} a_n (z - z_0)^n. \quad \text{Where } a_n = \overbrace{\frac{F^{(n)}(z_0)}{n!}}^{\text{exists!}} = \oint_\gamma \frac{\varphi(\zeta)}{(\zeta - z_0)^{n+1}} d\zeta. \text{ (Taylor series)}$$

Also $F(z) = F(z_0) + \frac{F'(z_0)}{1!}(z - z_0) + \dots + \frac{F^{(n-1)}(z_0)}{(n-1)!}(z - z_0)^{n-1} + (z - z_0)^n F_n(z)$,

$$F_n(z) := \oint_\gamma \frac{\varphi(\zeta)\, d\zeta}{(\zeta - z_0)^n (\zeta - z)}$$

(Taylor polynomial with Cauchy remainder).

Proof.

Observe:

$$\frac{1}{1-q} = 1 + q + \dots + q^{n-1} + \frac{q^n}{1-q} \quad (q \neq 1)$$

$$\frac{1}{1-q} = \sum_{n=0}^{\infty} q^n \quad (|q| < 1).$$

Fix $z_0 \notin \gamma$, z: $|z - z_0| < R = \operatorname{dist}(z_0, \gamma)$.

Cauchy trick: take $q = \frac{z - z_0}{\zeta - z_0}$, $\zeta \in \gamma$.

Then

$$\frac{1}{1-\frac{z-z_0}{\zeta-z_0}} = \sum_{k=0}^{n-1}\left(\frac{z-z_0}{\zeta-z_0}\right)^k + \frac{\left(\frac{z-z_0}{\zeta-z_0}\right)^n}{1-\frac{z-z_0}{\zeta-z_0}}$$

$$\frac{\zeta-z_0}{\zeta-z} = \sum_{k=0}^{n-1}\frac{(z-z_0)^k}{(\zeta-z_0)^k} + \frac{(z-z_0)^n}{(\zeta-z)(\zeta-z_0)^{n-1}}$$

$$\boxed{\frac{1}{\zeta-z} = \sum_{k=0}^{n-1}\frac{(z-z_0)^k}{(\zeta-z_0)^{k+1}} + \frac{(z-z_0)^n}{(\zeta-z)(\zeta-z_0)^n}}$$

Multiply by $\varphi(\zeta)$, $\oint_\gamma$:

$$F(z) = \oint_\gamma \frac{\varphi(\zeta)}{\zeta-z}\,d\zeta = \sum_{k=0}^{n-1}(z-z_0)^k \underbrace{\oint_\gamma \frac{\varphi(\zeta)}{(\zeta-z_0)^{k+1}}\,d\zeta}_{a_k} + (z-z_0)^n \underbrace{\oint_\gamma \frac{\varphi(\zeta)}{(\zeta-z_0)^n(\zeta-z)}\,d\zeta}_{F_n(z)}$$

Why is F differentiable and $a_n = \frac{F^{(n)}(z)}{n!}$?

For this, we show that

$F(z) = \sum_{n=0}^{\infty} a_n (z-z_0)^n$, if $|z-z_0|<R$. So the radius of convergence of $F(w+z_0) = \sum a_n w^n$ is at least R, and $\underline{\underline{a_n = \frac{F^{(n)}(z_0)}{n!}}}$.

By Cauchy Trick,

$$F(z) - \sum_{k=0}^{n-1} a_k (z-z_0)^k = \oint_\gamma \frac{\varphi(\zeta)}{(\zeta-z)(\zeta-z_0)} \cdot \frac{(z-z_0)^n}{(\zeta-z_0)^n}\,d\zeta = F_n(z)(z-z_0)^n$$

Let $M = \sup_\gamma |\varphi|$ ($<\infty$ - bounded!)

$$|(z-z_0)^n|\,|F_n(z)| \le \left(\frac{|z-z_0|}{R}\right)^n \cdot \frac{1}{R} \cdot \frac{M}{(R-|z-z_0|)} \cdot \ell(\gamma) \xrightarrow[n\to\infty]{} 0 \quad \left(\frac{|z-z_0|}{R} < 1\right)$$

$|\zeta - z_0| \ge R$ $\qquad$ $|\zeta - z| \ge |\zeta - z_0| - |z - z_0|$